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# Sharp Elements in Effect Algebras

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We show that if for an arbitrary pair of orthogonal sharp elements of an effect algebra $E$ its join exists and is sharp, then the set $E_S$ of all sharp elements of $E$ is a subeffect algebra of $E$ that is an orthomodular poset. Such effect algebras need not be sharply dominating but $S$-dominating. Further, we show that in every nonproper effect algebra $E$, $E_S$ is a subeffect algebra that is an orthomodular poset. Moreover, a general theorem for $E_S$ is proved.

## 1. INTRODUCTION AND BASIC DEFINITIONS AND FACTS

An effect algebra is a partial algebra that generalizes the set $\varepsilon(H)$ of positive self-adjoint operators on Hilbert space $H$ that are bounded above by the identity operator. Effect algebras were introduced by Foulis and Bennett (1994).

*Definition 1.1.* A structure $(E; \oplus, 0, 1)$ is called an effect-algebra if 0, 1 are two distinguished elements and $\oplus$ is a partially defined binary operation on $E$ that satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,
(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,
(Eiii) for every $a \in P$ there exists a unique $b \in P$ such that $a \oplus b = 1$ (we put $a' = b$),
(Eiv) if $1 \oplus a$ is defined then $a = 0$.

In every effect algebra $(E; \oplus, 0, 1)$ the partial binary operation $\ominus$ and the partial order $\leq$ can be defined by

$$a \leq c \text{ and } c \ominus a = b \text{ iff } a \oplus b \text{ is defined and } a \oplus b = c.$$

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If $E$ with the defined partial order is a lattice then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra*. Examples of lattice effect algebras are, for example, direct product or horizontal sum of an orthomodular lattice and *MV* algebra or horizontal sum of two *MV* algebras.

Moreland and Gudder (1999) noted that there are two main types of effects, the sharp effects that describe perfectly accurate yes–no measurements and the unsharp effects that describe imprecise yes–no measurements. However, the set of all sharp elements of an effect algebra $E$ need to be neither a subeffect algebra of $E$ nor an orthomodular lattice or poset (see Example 2.9). To remedy this shortcoming Gudder (1998a) introduced special types of effect algebras called sharply dominating and $S$-dominating.

*Definition 1.2* (Gudder, 1998a). Let $(E; \oplus, 0, 1)$ be an effect algebra.

(i) An element $w \in E$ is *sharp* if $w \wedge w' = 0$. Put $E_S = \{w \in E \mid w \wedge w' = 0\}$.
(ii) $E$ is *sharply dominating* if every $a \in E$ is dominating by a smallest sharp element $\hat{a} \in E_S$ (i.e., (i) $a \leq \hat{a}$, (ii) if $a \leq b \in E_S$, then $\hat{a} \leq b$).
(iii) $E$ is *S-dominating* if it is sharply dominating and $a \wedge w$ exists for every $a \in E, w \in E_S$.

In Gudder (1998a) it has been shown that in an $S$-dominating effect algebra $E$, $E_S$ forms an orthomodular lattice. Moreover, $\varepsilon(H)$ is $S$-dominating.

In Jenča and Riečanová (1999), it has been shown that in a lattice effect algebra $E$, $E_S$ is a subeffect algebra that is an orthomodular lattice.

In the present paper we show that in a nonproper effect algebra $E$, $E_S$ is a subeffect algebra that is an orthomodular poset.

*Definition 1.3.* An effect algebra $(E; \oplus, 0, 1)$ is called *proper* if there are $a, b \in E$ such that $a \leq b'$ and $a \wedge b$ exists but $a \vee b$ does not exist in $E$. $E$ is called *nonproper* if $E$ is not proper.

We can obtain *examples of effect algebras that are nonproper and simultaneously not lattice-ordered*, for example:

(1) When we consider orthomodular poset $(E; \leq, ', 0, 1)$ that is not lattice and say that for $a, b \in E, a \oplus b$ is defined iff $a \leq b'$, in which case $a \oplus b = a \vee b$.
(2) When we consider a direct product of two effect algebras $E_1 \times E_2$, where $E_1$ is associated to an orthomodular poset (not lattice, as described in (1)) and $E_2$ is a lattice effect algebra that is not an orthomodular lattice.

A lattice effect algebra that is not an orthomodular lattice and also not an *MV* effect algebra (*MV* algebra) is, for example, a direct product (or 0–1-pasting) of an orthomodular lattice and an *MV* algebra that are considered as two effect algebras. Obviously, this direct product (0–1-pasting) need not be a distributive lattice.

*Definition 1.4* (Greechie *et al.*, 1995). A subset $Q$ of an effect algebra $(E; \oplus, 0, 1)$ is called a *subeffect algebra* of $E$ if

(i) $0, 1 \in Q$,
(ii) $x \in Q \Longrightarrow x' \in Q$,
(iii) $(x, y \in Q$, with $x \leq y') \Longrightarrow x \oplus y \in Q$.

Evidently, subeffect algebra $Q$ of an effect algebra $E$ is an effect algebra in its own right under the restriction of $\oplus$ to $Q$. Then the partial order on $Q$ is a restriction to $Q$ of the partial order on $E$.

Recall, that $a, b \in E$ are called *orthogonal* if $a \leq b'$. Obviously, in effect algebra $E$, $a \oplus b$ is defined iff $a \leq b'$. If for elements $a \leq b'$ the $a \vee b$ exists, then $a \wedge b$ also exists and

$$a \oplus b = (a \vee b) \oplus (a \wedge b), \qquad \text{(see Greechie } et\ al.\text{, 1995)}$$

On the other hand if for $a \leq b'$ the $a \wedge b$ exists then $a \vee b$ need not exist (Example 2.14).

In Kôpka and Chovanec (1995, 1997), compatibility of two elements of an effect algebra $E$ was introduced. We say that $a, b \in E$ are *compatible* (written as $a \leftrightarrow b$) if there exist $u, v, w \in P$ such that $a = u \oplus w$, $b = v \oplus w$ and $u \oplus w \oplus v$ is defined. If $E$ is a lattice effect algebra then $a \leftrightarrow b$ iff $(a \vee b) \ominus a = b \ominus (a \wedge b)$. A lattice effect algebra in which every pair $a, b \in E$ is compatible is called a *Boolean effect algebra* (Riečanová, 2000b) or an *MV* effect algebra (Foulis in letter communications).

More Details on orthoposets, orthomodular posets, and orthomodular lattices can be found in Kalmbach (1983) and for more details on effect algebras in Foulis and Bennett (1994).

## 2. SHARP ELEMENTS IN EFFECT ALGEBRAS

For the remainder of this paper we assume that $(E; \oplus, 0, 1)$ is an effect algebra and $E_S = \{w \in E \mid w \wedge w' = 0\}$ is *the set of all sharp elements of* $E$. For $a, b \in E$ we denote $a \wedge_E b$ ($a \wedge_{E_S} b$), a meet of $a$ and $b$ in $E$ (in $E_S$). The meaning of $a \vee_E b$ ($a \vee_{E_S} b$) is dual.

**Theorem 2.1.** *Let $(E; \oplus, 0, 1)$ be an effect algebra. Then $E_S$ is an orthoposet that satisfies the following condition for all $w_1, w_2 \in E_S$:*

$$(w_1 \leq w_2 \textit{ and } w_1' \wedge_E w_2 = 0) \Longrightarrow (w_1 = w_2).$$

**Proof:** Assume that $w_1, w_2 \in E_S$ with $w_1 \leq w_2$. Then $w_1' \leftrightarrow w_2$, which implies that there are $u, w, v \in E$ such that $w_1' = u \oplus w$, $w_2 = w \oplus v$, and $u \oplus w \oplus v$

is defined. Since $w \leq w_1', w_2$, the assumption $w_1' \wedge_E w_2 = 0$ gives $w = 0$, and hence $w_1' = u$ and $w_2 = v$. It follows that $w_1' \oplus w_2$ is defined, which implies that $w_2 \leq w_1$. This proves that $w_1 = w_2$. □

**Corollary 2.2.** *Let in an effect algebra* $(E; \oplus, 0, 1)$ *for every pair* $w_1, w_2$ *of sharp elements,* $w_1 \wedge_E w_2$ *exists in* $E$ *and* $w_1 \wedge_E w_2 \in E_S$*. Then*

(i) $E_S$ *is a subeffect algebra of* $E$,
(ii) $E_S$ *is an orthomodular lattice.*

**Proof:** (i) Let $w_1, w_2 \in E_S$ with $w_1 \leq w_2'$. Then $w_1' \wedge_E w_2' \in E_S$, and $w_1 \wedge_E w_2 = 0$. It follows that $w_1 \oplus w_2 = w_1 \vee_E w_2 = (w_1' \wedge_E w_2')' \in E_S$. Since $E_S$ obviously is an ortholattice, the last implies that $E_S$ is a subeffect algebra of $E$ and, by Theorem 2.1, $E_S$ is an orthomodular lattice (see Kalmbach, 1983, p. 27). □

**Lemma 2.3.** *Let* $(E; \oplus, 0, 1)$ *be an S-dominating effect algebra. Then for all* $w_1, w_2 \in E_S$, $w_1 \vee_E w_2$ *and* $w_1 \wedge_E w_2$ *exist and they are sharp.*

**Proof:** Let $w_1, w_2 \in E_S$. Then $w_1 \wedge_E w_2$ exists in $E$ and hence there exists the smallest element $w \in E_S$ with $w_1 \wedge_E w_2 \leq w$. It follows that $w \leq w_1$ and $w \leq w_2$, which gives $w \leq w_1 \wedge_E w_2 \leq w$. Thus $w = w_1 \wedge_E w_2 \in E_S$. Moreover, $w_1', w_2' \in E_S$ and hence $w_1' \wedge_E w_2' \in E_S$, which gives $w_1 \vee_E w_2 \in E_S$. □

**Corollary 2.4** (Gudder, 1998a,b). *Let* $(E; \oplus, 0, 1)$ *be an S-dominating effect algebra. Then*

(i) $E_S$ *is a subeffect algebra of* $E$.
(ii) $E_S$ *is an orthomodular lattice.*

**Proposition 2.5.** *There are effect algebras that are not sharply dominating but joins and meets of two arbitrary sharp elements exist and they are sharp.*

*Example 2.6.* Let $E = \{0, a, b, a \oplus a, b \oplus b, a \oplus b, a', b', (a \oplus a)', (b \oplus b)', (a \oplus b)', 1\}$ be an effect algebra in which $a, b, (a \oplus a)', (b \oplus b)', (a \oplus b)'$ are atoms (hence $a', b', a \oplus a, b \oplus b, a \oplus b$ are coatoms). Moreover, $a' = a \oplus (a \oplus a)' = b \oplus (a \oplus b)'$ and $b' = a \oplus (a \oplus b)' = b \oplus (b \oplus b)'$. Further for every $x \in E$, $x \oplus x' = 1$ and $0 \oplus x = x$.
Then

- $E$ is a proper effect algebra because $a \wedge b = 0$; $a \leq b'$ and $a \vee b$ does not exist.
- $E_S = \{a \oplus a, b \oplus b, a \oplus b, (a \oplus a)', (b \oplus b)', (a \oplus b)', 0, 1\}$.

- Joins and meets of arbitrary two sharp elements exist in $E$ and they are sharp.
- $E_S$ is a subeffect algebra of $E$ that is an orthomodular lattice.
- $E_S$ is not sharply dominating ($S$-dominating); because for element $a \in E$ there does not exist a smallest sharp element $w \in E_S$ with $a \leq w$.

Note that *every complete effect algebra* $(E; \oplus, 0, 1)$ (i.e., $E$ is a complete lattice) *is S-dominating*. This follows from the fact that in every complete effect algebra $E$, $E_S$ is a complete orthomodular lattice (see Jenča and Riečanová, 1999). Effect algebras that can be densely embedded into complete effect algebras have been characterized in Riečanová (2000a). Such effect algebras are nonproper (Riečanová, 2000a, Example 6.1). Thus: *If E can be densely embedded into a complete effect algebra, then* $E_S$ *is a subeffect algebra of E that is an orthomodular poset* (Theorem 2.10 of this paper).

**Theorem 2.7.** *Let* $(E; \oplus, 0, 1)$ *be an effect algebra in which for all* $w_1, w_2 \in E_S$ *with* $w_1 \leq w_2'$ *there exists* $w_1 \vee_E w_2$ *and* $w_1 \vee_E w_2 \in E_S$. *Then*

(i) $E_S$ *is a subeffect algebra of E.*
(ii) $E_S$ *is an orthomodular poset.*

**Proof:**

(i) If $w_1, w_2 \in E_S$ with $w_1 \leq w_2'$ then $w_1 \wedge_E w_2 = 0$ and $w_1 \oplus w_2 = w_1 \vee_E w_2 \in E_S$. Since $0, 1 \in E_S$, and $w \in E_S$ iff $w' \in E_S$, we conclude that $E_S$ is a subeffect algebra of $E$.
(ii) Assume that $v_1, v_2 \in E_S$ with $v_1 \leq v_2$. It gives that $v_1 \leq (v_2')'$, which implies that $v_1 \vee_E v_2' \in E_S$ and hence also $v_1' \wedge_E v_2 \in E_S$. It follows that $v_1' \wedge_{E_S} v_2 = v_1' \wedge_E v_2 = 0$, which by Theorem 2.1 gives $v_1 = v_2$. This proves that $E_S$ is an orthomodular poset (see Kalmbach, 1983, p. 27, Theorem 11). □

**Proposition 2.8.** *There are proper effect algebras in which there exist* $x, y \in E_S$, *with* $x \wedge_E y \notin E_S$ *and* $x \wedge_{E_S} y \neq x \wedge_E y$.

*Example 2.9.* Let $E = \{0, a, b, a \oplus a, a \oplus b, a', b', (a \oplus a)', (a \oplus b)', 1\}$ be an effect algebra in which $a, b, (a \oplus a)', (a \oplus b)'$ are atoms (hence $a', b', a \oplus a, a \oplus b$ are coatoms), and $a' = a \oplus (a \oplus a)' = b \oplus (a \oplus b)'$ and $b' = a \oplus (a \oplus b)'$. Moreover, for every $x \in E$, $x \oplus x' = 1$ and $x \oplus 0 = x$.

Obviously, $E_S$ is not a subeffect algebra of $E$; because $b, (a \oplus b)' \in E_S$ but $a' = b \oplus (a \oplus b)' \notin E_S$. Moreover, $E_S$ is not an orthomodular poset since $b \leq a \oplus b$ and $b \neq (a \oplus b)$, but $b' \wedge_{E_S} (a \oplus b) = 0$. Further $b' \wedge_E (a \oplus b) = a \neq$

$b' \wedge_{E_S} (a \oplus b)$. Finally, $E$ is proper since $a \leq b'$ and $a \wedge_E b = 0$, but $a \vee_E b$ does not exist.

**Theorem 2.10.** *Let $(E; \oplus, 0, 1)$ be a nonproper effect algebra. Then*

(i) *If $w_1, w_2 \in E_S$ and $w_1 \vee_E w_2$ exists in $E$, then $w_1 \vee_E w_2 \in E_S$.*
(ii) *If $w_1, w_2 \in E_S$ and $w_1 \wedge_E w_2$ exists in $E$, then $w_1 \wedge_E w_2 \in E_S$.*
(iii) *$E_S$ is a subeffect algebra of $E$.*
(iv) *$E_S$ is an orthomodular poset.*

**Proof:**

(i) Let $w_1, w_2 \in E_S$ and let $w_1 \vee_E w_2$ exists in $E$. Then $w_1' \wedge_E w_2'$ exists in $E$. Since $E$ is nonproper, $w_1 \oplus (w_1' \wedge_E w_2') = w_1 \vee_E (w_1' \wedge_E w_2')$ and $w_2 \oplus (w_1' \wedge_E w_2') = w_2 \vee_E (w_1' \wedge_E w_2')$. Assume that $d \in E$ with $w_1 \vee_E (w_1' \wedge_E w_2') \leq d$ and $w_2 \vee_E (w_1' \wedge_E w_2') \leq d$. Then $w_1, w_2 \leq d \ominus (w_1' \wedge_E w_2')$, and we obtain $(w_1 \vee_E w_2) \oplus (w_1' \wedge_E w_2') \leq d$. The last condition implies that $d = 1$, hence $(w_1 \vee_E (w_1' \wedge_E w_2') \vee_E (w_2 \vee_E (w_1' \wedge_E w_2')' = 1$. Since $w_1 \vee_E w_2$ exists, we conclude that $(w_1 \vee_E w_2) \vee (w_1' \wedge_E w_2') = 1$, which gives that $w_1 \vee_E w_2 \in E_S$.
(ii) It follows by (i) and d'Morgan laws.
(iii), (iv) Let $w_1, w_2 \in E_S$ with $w_1 \leq w_2'$. Since $E$ is not proper and $w_1 \wedge_E w_2 = 0$ we conclude that $w_1 \vee_E w_2$ exists in $E$. By (i), $w_1 \vee_E w_2 \in E_S$. By Theorem 2.7, $E_S$ is a subeffect algebra of $E$ that is an orthomodular poset. □

**Corollary 2.11.** *A nonproper effect algebra $(E; \oplus, 0, 1)$ is an orthomodular poset iff every element of $E$ is sharp.*

**Corollary 2.12** (Jenča and Riečanová, 1999). *In every lattice effect algebra $E$, $E_S$ is a subeffect algebra and a sublattice that is an orthomodular lattice.*

**Proposition 2.13.** *There are proper effect algebras in which $E_S$ is a subeffect algebra that is an orthomodular lattice (Boolean algebra).*

*Example 2.14.* Let $E = \{0, a, b, a', b', 1\}$ be an effect algebra in which $a' = a \oplus b$, $b' = a \oplus a = b \oplus b$ and $1 = x \oplus x'$, $x = 0 \oplus x$ for every $x \in E$. $E$ is proper because $a \leq b'$ and $a \wedge b = 0$ but $a \vee b$ does not exist. Obviously $E_S = \{0, 1\}$.

For an effect algebra $(E; \oplus, 0, 1)$, the set $B_E = \{y \in E | \leftrightarrow x$ for all $x \in E\}$ is called a *compatibility center* of $E$ and the set $C(E) = \{z \in E | x = (x \wedge z) \vee$

$(x \wedge z')$ for all $x \in E\}$ is called a *center* of $E$. It was shown in Riečanová (1999a,b) that in every effect algebra $E$ we have $C(E) \subseteq B_E \cap E_S$.

**Theorem 2.15.** *Let* $(E; \oplus, 0, 1)$ *be a nonproper effect algebra in which* $x \wedge_E z$ *exists for all* $x \in E$, $z \in E_S$. *Then* $C(E) = B_E \cap E_S$.

**Proof:** Let $z \in B_E \cap E_S$ and $x \in E$. Then $z \wedge z' = 0$ and $z \leftrightarrow x$. It follows that there are $u, v, w \in E$ such that $x = u \oplus w$, $z = w \oplus v$ and $u \oplus w \oplus v$ is defined in $E$. Let $y \in E$ be such that $u \oplus w \oplus v \oplus y = 1$, which gives $z' = u \oplus y$. It follows that $u \leq x \wedge z'$ and $w \leq x \wedge z$ hence $u \wedge w = 0$. Thus $x = u \oplus w = u \vee w \leq (x \wedge z') \vee (x \wedge z) \leq x$. We conclude that for all $x \in E$, $x = (x \wedge z) \vee (x \wedge z')$, which prove that $z \in C(E)$. Since $C(E) \subseteq B_E \cap E_S$, we conclude that $C(E) = B_E \cap E_S$. $\square$

A lattice effect algebra $(E; \oplus, 0, 1)$ in which every two elements are compatible is called a Boolean effect algebra (Riečanová, 2000b) or also *MV* effect algebra (Foulis in letter communications). For a Boolean effect algebra $E$ we have $E = B_E$ and thus $C(E) = E_S$, in view of Theorem 2.15—but not conversely. For example, for the proper effect algebra $E = \{0, a, b, a', b', 1\}$ in Example 2.14 we have $E = B_E$ and $C(E) = E_S = \{0, 1\}$.

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